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UDC 534

ON THE PRINCIPAL RESONANCE IN MECHANICS OF CONTINUOUS MEDIA

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Small amplitude resonance oscillations in a nonlinear system close to the stability limit are considered. The Van der Pol equation with a supplementary parameter is derived for the oscillation amplitude; in an autonomous system that parameter defines the dependence of oscillation frequency on amplitude.

Development of stationary motions with parameter variation is investigated using the amplitude equation. Resonance acoustic oscillations of a polytropic gas, induced by a piston in a long pipe with a chamber at its end, are considered as an example.

1. Resonance phenomena are possible under periodic action on systems defined by equations in partial derivatives. Examples of resonance are well known in the theory of elasticity, acoustics and electrodynamics. Resonance in self-excited systems may induce modes of beat and of forced oscillations; resonance in a gas discharge which is unstable with respect to stria excitation provides an example of this [1].

Here we consider resonance oscillations of small amplitude for which deviations of dependent variables (density, velocities, current, etc.) from their equilibrium values independent of time are small. The derivations can be small, if the amplitude ε of the periodic perturbation of the autonomous system are small.

Similarly to the autonomous systems considered in [2], the problem for the deviation vector X is derived by expanding the input nonlinear equations and boundary conditions in series in powers of X and ε The simplest nonautonomous equation linear with respect to ε is

$$\partial X / \partial t + L_1 X + L_2 X^2 + L_3 X^3 + \ldots = \varepsilon E C + c.\ c., \quad E = e^{i\omega t} \quad (1.1)$$

Here and subsequently c. c. denotes an expression that is complex conjugate of the preceding one. Coefficients $L = L(D, x, \lambda)$, where λ are parameters of the system, x are space coordinates, and $D = \partial / \partial x$ are operators of differentiation; coefficients L real with respect to D are polynomials. Terms in the right-hand part of (1.1) define perturbations of complex amplitude ε of a form defined by the complex vector $C = C(x)$.

It is assumed that the region of variation of x is bounded and the boundary conditions are of the simplest form: if x belongs to the region boundary, $UX = 0$, where $U = U(D, x, \lambda)$, is a real matrix. The general case when equations and boundary conditions contain higher powers of ε, X and $\partial / \partial t$, is considered in Sect. 4.

Since stationary (or close to these) states of the system are considered below, the boundary value problem is solved without initial conditions.

For $\varepsilon = 0$ Eq. (1.1) is autonomous. Solutions of the autonomous problem linear with respect to X are determined by equations

$$X = X_1(x) e^{p_0 t}, \quad p_0 X_1 + L_1 X_1 = 0, \quad UX_1 = 0 \quad (1.2)$$

If X is a solution of problem (1.2), then $\overline{X}$ is also a solution, hence we consider below eigenvalues $p_0 = \gamma_0 + i\Omega_0$ with frequencies $\Omega_0 \geqslant 0$.

It is assumed that critical values of the system parameters $\lambda = \lambda_*$ exist for which one of the eigenvalues (below called critical and assumed to be simple) becomes pure-imaginary ($p_0 = i\Omega_*$), and that the increments $\gamma_0(\lambda_*)$ of all other eigenvalues are negative and not small.

The principal resonance occurs for $\varepsilon \neq 0$, if the frequency difference $p = p_0 - i\omega$ is small, i. e. the increment γ_0 and the real frequency difference $\Delta = \Omega_0 - \omega$ are small (in comparison with ω). The perturbation amplitude ε is also to be considered small so that the quantity X could be small for considerable t. The small parameters γ_0, Δ and ε are obviously independent, their ratios can be arbitrary.

The method of finding stationary oscillations in autonomous systems in [2] is extended to the case of nonautonomous systems, as is done in the theory of oscillations [3, 4]. The solution is sought in the form of series in powers of the small quantities QE, εE, and of their complex conjugate

$$X = [E\,(QX_1 + Q^2\overline{Q}X_3 + \varepsilon X_4) + (EQ)^2 X_2] + \text{c.c.} + Q\overline{Q}X_0 + \ldots \quad (1.3)$$

The series coefficients X_n depend only on x, and X_1 is the eigen function of problem (1.2) which corresponds to the critical p_0 (with frequency $\Omega_0 > 0$). The equation for the oscillation amplitude $Q\ (t)$ is also sought in the form of series in Q and ε and their conjugate

$$dQ\ /\ dt = Q\,(p + p_2 Q\overline{Q}) + \varepsilon p_1 + \ldots \quad (p = p_0 - i\omega) \quad (1.4)$$

As in the case of autonomous systems [2], the coefficients in (1.3) are successively determined from linear inhomogeneous boundary value problems; the conditions of coefficient boundedness for $p \to 0$ determine the coefficients in (1.4).

Inhomogeneous problems are obtained after the substitution of (1.3) and (1.4) into (1.1) and the boundary conditions are derived by equating terms with like powers of E, Q, $\overline{Q}$, ε and $\bar{\varepsilon}$. It is sufficient to consider problems that correspond to nonnegative powers ν of quantity E. For $\nu \neq 1$ solutions of such problems are bounded, while for $\nu = 1$ they are finite for $p \to 0$ only if

$$\langle \Psi \cdot Z \rangle \equiv \int (\Psi \cdot \overline{Z})\, dx = 0 \quad (1.5)$$

where Ψ is the free term of the inhomogeneous problem, Z is the eigen function of the problem conjugate of problem (1.2) with the critical value p_0, and integration is carried out over the region of variation of x. The coefficient p_n of series (1.4) at powers of Q, $\overline{Q}$, ε and $\bar{\varepsilon}$, appearing linearly in Ψ and determining the inhomogeneous problem, is obtained from (1.5) (e.g. p_2 is determined from the problem for X_3 and p_1 from that for X_4).

In what follows the system behavior is investigated on the basis of the approximate equation (1.4). To make such approximation possible it is necessary for the retained explicitly written terms to be the highest. The latter takes place, when for $\lambda = \lambda_*$ the quantities $p_n = \gamma_n + i\Omega_n$ $(n = 1,2)$ are not small; it is assumed that these conditions are satisfied. In investigations of stability of stationary oscillations it is also necessary to assume that γ_2 is not small (as in the case of $\varepsilon = 0$).

To determine p_n it is necessary to retain in (1.3), (1.1) the explicitly written terms. In the approximation used here it is sufficient to determine the quantities p_n and X_n for $\lambda = \lambda_*$ and $\omega = \Omega_*$. It is convenient to determine $p_{1,2}$ and $X_{3,4}$ simultaneously, setting

$$Y = X_3 + gX_4, \quad P = p_2 + gp_1, \quad g = \varepsilon\ /\ (Q^2\overline{Q}) \quad (1.6)$$

and considering the particular case of $g \sim 1$ (here and subsequently the relation $a \sim b$ is equivalent to $a = O\ (b)$; in these relationships and inequalities complex numbers are equal in modulo). It may be more convenient in practical applications to determine the coefficients in (1.3) and (1.4) in the form of series in $\lambda - \lambda_*$ and $\omega - \Omega_*$.

An example of the computation of P is given in Sect. 5.

Generalizations of the nonautonomous problem (1.1) are considered in Sect. 4, where the complete series (1.3) and (1.4) are also presented.

2. Periodic oscillations of the system correspond to stationary solutions (1.4), while nearly-periodic oscillations (beats) correspond to periodic solutions. Observed oscillations correspond to stable solutions. Stationary solutions $(\partial / \partial t = 0)$ are considered in this Section.

The estimate $Q \sim \min (\varepsilon^{1/3}, \varepsilon / p)$ can be readily obtained from (1.4). A linear resonance occurs for $\varepsilon^{2/3} \ll p$, since nonlinear effects are unimportant for the determination of the stationary amplitude; in the opposite case the effects of frequency differences become unimportant.

To determine the exact stationary solutions it is convenient to present (1.4) in the form

$$dR / d\tau = (1 + i\eta)\{R\ [\alpha (1 - i\sigma) - R\bar{R}] + F + \ldots\} \equiv H \tag{2.1}$$

$$R = \frac{Q}{\mu} e^{i(\varphi_2 - \varphi_1)}, \quad \tau = t \,|\, p_2 \,|\, \mu^2 \cos \varphi_2, \quad \eta = \operatorname{tg} \varphi_2 = \frac{\Omega_2}{\gamma_2}$$

$$\alpha = \frac{\cos \psi}{|\cos \psi|} = \pm 1, \quad \sigma = \operatorname{tg} \psi, \quad \mu = \left| \frac{p}{p_2} \cos \psi \right|^{1/2} > 0, \quad \psi = \varphi_2 - \varphi$$

$$F = \left| \varepsilon \frac{p_1}{p_2} \right| \mu^{-3}, \quad p = |p| e^{i\varphi}, \quad p_2 = -|p_2| e^{i\varphi_2}, \quad \varepsilon p_1 = |\varepsilon p_1| e^{i\varphi_1}$$

The Van der Pol equations analyzed in [3, 5 — 7] are obtained from (2.1) for $\eta = 0$, $\alpha = 1$ and $R = x + iy = re^{i\theta}$ Equations that reduce to (2.1) were derived and analyzed in [8]. Results of a qualitative investigation of (2.1) and those obtained in [8] are presented below.

First, the case of $\gamma_2 < 0$ and $\eta \geqslant 0$ is considered. For stationary solutions (2.1) yields

$$\rho\ [(\alpha - p)^2 + \sigma^2] = f, \quad \rho = R\bar{R}, \ f = F^2 \tag{2.2}$$

The reversal of signs of ρ and f in (2.2) for $\alpha = -1$ yields Eq. (2.2) for $\alpha = 1$. Instead of two equations (2.2) it is convenient to consider for $\rho \geqslant 0$ only the equation with $\alpha = 1$ in region $(-\infty < \rho < \infty)$ using negative ρ and f for $\alpha = -1$.

The amplitude curves $f(\sigma, \rho) = \text{const}$ were constructed in [5]. In Fig. 1 continuous thin lines relate to curves $f = (-8, -4, 0{,}2, 4{,}8) / 27$, and the heavy dash lines relate to the ellipse $\partial f / \partial p \equiv (1 - \rho)(1 - 3\rho) + \sigma^2 = 0$. The amplitude curves at points of intersection with the ellipse are vertical.

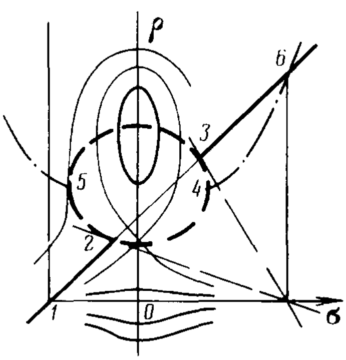

Fig. 1

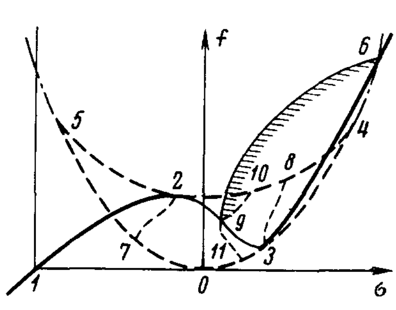

Fig. 2

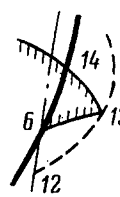

Fig. 3

Small deviations from equilibrium are proportional to $\exp(\varkappa t)$. From (2.1) we obtain

$$\varkappa = a \pm (a^2 - b^2)^{1/2}, \quad a \equiv \operatorname{Re}(\partial H / \partial R) = \alpha (1 + \sigma\eta - 2\rho)$$
$$b \equiv |\,\partial H / \partial R\,|^2 - |\,\partial H / \partial \bar{R}\,|^2 = (1 + \eta^2)\, \partial f / \partial \rho \tag{2.3}$$

where the derivatives are determined at the equilibrium point. It follows from (2.3) that inside the ellipse lie saddle points.

The straight line $a = 0$ intersects the σ-axis at point $\sigma_1 = -1/\eta$ and the ellipse at points *2* and *3*. Fig. 1 relates to $\eta > 1/\sqrt{3} = \eta_*$ when point *4* lies under the straight line.

Thin dash straight lines in Fig. 1 are tangent to the ellipse at points *2* and *3* and intersect the σ-axis at point $\sigma = \eta$; nodal points lie between these straight lines. Points of the ellipse are saddle-nodes (except the degenerate (angle) points *2* and *3*, and nodal points *4* and *5*). The remaining singular points are focal points.

Nodal and focal points are unstable if they lie between the straight line $a = 0$ and the σ-axis in the first and third quadrants formed by the σ-axis and the straight line $\sigma = \sigma_1$ which corresponds to the stability boundary $\gamma_0 = 0$. The second and fourth quadrants correspond to $\gamma_0 < 0$. For a continuous variation of parameters the transition from the upper half-plane to the lower (and conversely) in Fig. 1 occurs only through an infinitely distant point and into the opposite quadrant.

It can be shown with the use of methods [2, 9] that for $\delta \neq \eta$, $a = 0$ are compound focal points of first multiplicity; those lying above the ellipse or the parabola $2\rho = 1 + \sigma^2$ (the dash-dot line in Fig. 1) are unstable. The limit cycle $R(t)$ around focus R with a small a is

$$R(t) = R + (R / |R|)(x + iy)$$

$$(x, y) = {}^1/_2 S e^{i\Omega t} [ib^{1/2} - \alpha\rho, \alpha d] + cc + O(S^2)$$

$$d = \eta - \sigma - 3\eta\rho, \quad \Omega^2 = b + O(S^2), \quad |S|^2 = \frac{a}{d}\frac{\rho}{f}\frac{b}{(1+\eta^2)(\eta-\sigma)}$$

where a and b are determined in (2.3). An unstable cycle exists for $a < 0$, $\sigma < \eta$ and $d < 0$, and a stable one for $a > 0$ and $d(\sigma - \eta) < 0$.

Investigation of the compound focus *6* at the parabola is difficult owing to the awkwardness of related computations. It is reasonable to assume that this focus is of second multiplicity for any η, except the possible critical values at passage through which the focus changes stability.

3. *Knowing the nature of singular points it is possible to determine the structure of* the phase plane of Eq. (2.1) for various values of parameters σ and f.

The pattern of the subdivision of the parameter plane (Fig. 2) into regions of invariable (or slightly variable) structure is considered below. When that subdivision is known, the behavior of the system under parameter variation is determined elementary.

The similarly drawn curves and points denoted by the same numerals in Fig. 2 correspond to curves and points in Fig. 1 (owing to the presence of several singular points the reverse relationship is ambiguous).

The plane in Fig. 2 is divided by the δ-axis and the straight line $\delta = \delta_1$ into four quadrants. In the second and fourth quadrants $a = \alpha(1 + \sigma\eta) - 2R\bar{R} \leqslant 0$ for any R, hence according to the Bendickson criterion there are no limit cycles. The integral curves run from an unstable infinitely distant point to one of the stable points (there can be only two such points in the second quadrant for $\eta > \sqrt{3}$ when $\sigma_5 < \delta_1$). The same structure appears in a part of the third quadrant where the focal point is stable. If the focal point is unstable, there exists a limit cycle generated by the compound focal point at loss of stability.

A stable cycle exists in the first quadrant between the σ-axis and the curve *1, 2, 7, 0,* *11*, *9*, *6*, ∞. At intersection of curves *1*, *2* and 6, ∞ the cycle vanishes by contracting into the focal point. At intersection 2, 7 (9, 11) the cycle vanishes by merging with the saddle point separatrix which lies outside (inside) the cycle. At intersection *7*, *0*, *11* the cycle vanishes because of the appearance on it of a saddle-node, and at intersection 9, 6 the cycle vanishes by merging with the linear inner unstable cycle,

An unstable cycle exists in region 3, 8, 10, 9, 6, 3. At intersection 3, 6 it contracts into a point. At intersection 3, 8 (9, 10) it merges outside (inside) the cycle with the saddle point separatrix, and at the intersection 8, 10 a saddle-node appears in the cycle.

Point 9 lies on curve 2, 3 (a saddle point on the straight line 2, 3 corresponds to it in Fig. 1).

When η decreases to η_* the region of unstable cycle vanishes by contracting to point 4. For $\sigma > 0$ and $\eta_* > \eta > 0$ regions of invariant structure are the same as in Fig. 2 for $\sigma < 0$. For $\eta = 0$ Fig. 2 is symmetric about axis f and according to [7] $f_7 > f_2$.

A change of η results in a turn of the direction field of Eq. (2. 1) by a constant angle. The results obtained in [9] show that in this case the double cycle splits with increasing η into two, i. e. the curve 6, 9 recedes from point 4. This and the absence of cycles in the second quadrant imply that the curve 6, 9 lies in the region $\sigma > 0$.

The subdivision shown in Fig. 2 was obtained in [8]; it corresponds to a stable focal point 6. When η passes through certain values a change of the focal point stability is possible. Let η_{12} be one of such values with the focal point 6 unstable for $\eta > \eta_{12}$ and stable for $\eta \leqslant \eta_{12}$, and then the focal point neighborhood is of the form shown in Fig. 3. Three cycles exist on curve 6, 13, 14 in region 6, 13, 14 one of which is double, and at point 13 there is a triple cycle. With increasing η curve 6, 13 recedes from point 4.

The case of $\gamma_2 < 0$ and $\eta \geqslant 0$ was considered above; that of $\gamma_2 < 0$ and $\eta \leqslant 0$ needs no explanation, and that of $\gamma_2 = 0$ is considered below. The case of $\gamma_2 > 0$ differs from the considered by the direction of trajectories of Eq. (2. 1).

Note the relation between the resonance oscillation pattern and the stability index of an autonomous system. If $\gamma_0 \leqslant 0$, then for $\gamma_2 < 0$ the oscillations are periodic and if $\gamma_0 > 0$, then beats are also possible. Small amplitude oscillations are not possible for $\gamma_2 > 0$ and $\gamma_0 \geqslant 0$, when $\gamma_0 < 0$, then small oscillations (either periodic or beat) are possible.

4. Generalizations of problem (1. 1) and cases of violation of some of the limitations introduced above are considered in this Section.

It was assumed above that γ_2 is not small. If this is not so, the stationary solution Q (as before, determined by (2. 1) and (2. 2)) is stable if $\gamma_0 < \gamma_* < 0$, where $\gamma_* \sim Q\varepsilon$. These conditions are associated with the effect of terms $\sim Q^2\varepsilon$ and Q^5 in (1.4) on stability.

If p_n in (1. 4) are small, it is necessary to retain the disregarded terms of series (1. 3) and (1. 4). These series are of the form

$$X = \sum_{\nu=-\infty}^{\infty} X_\nu E^\nu \tag{4. 1}$$

$$\overline{X}_{-\nu} = X_\nu = \sum_{n=0}^{\infty} \sum_{r=0}^{n+\nu} \sum_{s=0}^{n} Q^{\nu+n-r} \overline{Q}^{n-s} \varepsilon^r \overline{\varepsilon}^s X_{\nu nrs} \qquad (\nu \geqslant 0)$$

$$\frac{dQ}{dt} = \sum_{n=0}^{\infty} \sum_{r=0}^{n+1} \sum_{s=0}^{n} Q^{1+n-r} \overline{Q}^{n-s} \varepsilon^r \overline{\varepsilon}^s p_{nrs}$$

Coefficients p are determined by the condition of boundedness of coefficients in X_1 for small frequency differences. In the case of considerable real frequency differences Eq. (4. 1), although usable, is ineffective, since it necessitates additional transformations. In the case of problem (1. 1) with initial condition $X(0) = X_+$ (where X_+ is small) it can be expected that Eq. (4. 1) defines the system behavior beginning at $t \sim 1 / \gamma_*$ (γ_* is the minimum decrement of noncritical eigenvalues of problem (1. 2)), when according to the linear theory only oscillations defined by the critical value p_0 are important in the system. Hence it is possible to take as the initial condition for (4. 1) $Q(0) = Q_+ = \langle X_+ \cdot Z\rangle / \langle X_1 \cdot Z\rangle$, where the notation is the same as in (1. 5). Expansions (4. 1) are also applicable in boundary value problems with nonlinear nonautonomous boundary conditions. These equations and conditions may contain higher order derivatives with respect to t and nonautonomous terms dependent on X; the latter may be nonlinear with respect to ε and nonharmonic with respect to explicit t. For the applicability of approximation (1. 4) it is necessary for the free term $\sim \varepsilon$ in the equation and in the boundary condition to be harmonic; if it is periodic with slowly decreasing harmonics, then (1. 4) is applicable for $\varepsilon \ll Q^2$ owing to the effect of harmonics with frequencies $\omega = 0$ and $2\Omega_0$.

It was assumed above that problem (1. 2) has only one critical p_0. Let p_3 be another such value (γ_3 is small); then there exist two relatively prime numbers m and n for which $p_3 \approx (n / m)\, p_0$, i. e. there is internal resonance. By jointly considering the equations for Q and Q_3 it is possible to find that approximation (1. 4) is applicable to Q if $m + n > 4$, $\gamma_3 < \gamma_* < 0$, where $\gamma_* \sim Q^2 \sim \varepsilon^{2/3}$; when the perturbation is nonharmonic, there must be no principal resonance of frequency Ω_3.

We note in conclusion that all of the above applies also to systems defined by ordinary differential equations.

5. As an example let us consider the problem for $X = (\xi, w)$

$$\begin{aligned} &\xi^{\cdot} + w' = 0, \quad w^{\cdot} + \varepsilon' + \Phi' + \lambda w = 0 \\ &\Phi = w^2 / \rho - \rho + \rho^\beta / \beta = w^2 (1 - \xi) + \xi^2 (h + h_* \xi) + \ldots \\ &h = {}^1/_2 (\beta - 1), \quad h_* = {}^1/_3\, h \ (\beta = 2); \quad 0 \leqslant x \leqslant l = 1 + 2 \quad (\varepsilon / \omega) \sin \omega t \\ &(a\xi^{\cdot} + w)_0 = 0 \quad (w)_1 = 2\, (\rho)_1\, \varepsilon \cos \omega t \end{aligned} \tag{5. 1}$$

This problem defines oscillations of gas in a pipe with a chamber, that are induced by an impermeable piston moving at the opposite end of the pipe at velocity $2\varepsilon \cos \omega t$. The problem is defined in dimensionless form such that for $\varepsilon = 0$ the pipe length, the speed of sound, and the density of gas in equilibrium are equal unity; $\xi = \rho - 1$ is the density deviation, $w = \rho v$ (where v is the velocity), λw is the friction force at the wall; parameter λ is proportional to the dimensionless coefficient of kinematic viscosity divided by the square of the pipe radius; a is the ratio of volumes of chamber and pipe, and the pressure of gas is ρ^β. The dot and prime denote differentiation with respect to t and x, respectively; the subscripts 0 and 1 outside parentheses in the boundary conditions relate to $x = 0$ and l, respectively. The first condition relates to the case when the chamber size is small in comparison with $1/\omega$ and, consequently, the dependence of pressure in the chamber on coordinates is negligible.

The condition of total mass of gas conservation

$$\int_0^l \xi dx + \alpha (\xi)_0 = 0 \tag{5. 2}$$

is used below. It is obtained by integrating the equation of continuity from $x = 0$ to $x = l + 0$ with allowance for boundary conditions and equalities $\rho = w = 0$ for $x > l$.

Solution of the linear autonomous problem (5. 1) is determined by the relationships

$$p_0 = ik, \quad k = {}^1/_2\, i\lambda + (\varkappa^2 - {}^1/_4\, \lambda^2)^{1/2}, \quad \mathrm{tg}\, \varkappa = -a\varkappa \tag{5.3}$$

$$X_1 = \left(\cos\theta, -i\frac{k}{\varkappa}\sin\theta\right), \quad \theta = \varkappa(x-1), \quad \varkappa \geqslant 0, \quad 0 \leqslant x \leqslant 1$$

The eigen function Z of the conjugate problem is obtained from X_1 by the substitution $k \to \bar{k}$; below we use the values of eigen functions for $\lambda = 0$, when $Z = X_1$ and $\langle X_1 \cdot Z\rangle = 1$.

In what follows frequency ω is assumed to be close to the first root $\varkappa = \varkappa_1$ of Eq.(5.3). The value of a is selected so that internal resonances do not violate approximation (1.4); such values exist (e.g. $|m\varkappa_s - n\varkappa| > {}^1/_3$, if $m + n < 10$, $a = 0.81$, $s > 1$). The number $k = \varkappa = 0$ and functions $X_1 = Z = (1,0)$ (they only define density variations) are also eigenvalues and eigen functions, which does not lead to any difficulties, since variations of density satisfy condition (5. 2).

Terms $\sim Q\varepsilon$ are not taken into account in approximation (1. 4). It is possible to assume that in the second boundary condition (5. 1) $(\rho)_1 = 1$ which is accurate to within these terms; the same accuracy obtains when $l = 1$ is assumed (this can be readily ascertained by introducing the new coordinate $y = x / l$).

For $\lambda = 0$ and $\omega = \varkappa$ from (5. 1) we obtain for X_n in (1. 3)

$$in\varkappa\xi + w' = 0, \quad in\varkappa w + \xi' + \Phi_n' = 0$$
$$(in\varkappa\xi + w)_0 = 0 \quad (w)_1 = 0 \quad (n = 0,2)$$
$$\Phi_2 = w_1^2 + h\xi_1^2, \quad \Phi_0 = 2\,(|w_1|^2 + h\,|\xi_1|^2)$$

The solutions of these problems are

$$w_0 = 0, \ \xi_0 = (h-1)\ (c_0 + \cos 2\theta), \quad c_0 = (\sin^2\varkappa) / (1 + 1/a) \tag{5.4}$$
$$w_2 = {}^1/_4 i\,(1+h)\,(2\theta\cos 2\theta + c_2\sin 2\theta), \quad \xi_2 = 1/_4\ (1+h) \times$$
$$[2\theta\sin 2\theta - (1 + c_2)\cos 2\theta], \quad c_2 = (1 + 1/a)\ (a\varkappa)^{-2} + 3/a - 1$$

where the constant c_0 is determined by condition (5. 2) for ξ_0. The equality $w_0 = 0$ means that in the course of establishment of oscillations the distribution ξ_0 is generated by the mean stream $w_0 = 0\,(Q^2)$.

For Y in (1. 6) we obtain

$$i\varkappa\xi + w' + P\xi_1 = 0, \quad i\varkappa w + \xi' + Pw_1 + \Phi_3' = 0 \tag{5.5}$$
$$a\,(i\varkappa\xi + P\xi_1)_0 + (w_0) = 0, \quad (w)_1 = g$$
$$\Phi_3 = 2w_1\bar{w}_1 - w_1^2\bar{\xi}_1 - 2\,|w_1|^2\,\xi_1 + 2h\ (\xi_0\xi_1 + \xi_2\bar{\xi}_1) + 3h_*\xi_1^2\bar{\xi}_1$$

The substitution

$$Y = Y_* + A / (i\varkappa), \quad A = (-g/a - P\cos\ ,\ i\varkappa g)$$

reduces (5. 5) to the problem for Y_* with homogeneous conditions and the free term $\psi = A + (o,\ \Phi_3') + PX_1$. A solution of this problem exists if $\langle\psi\cdot Z\rangle = 0$; this yields

$$P = -(g + iJ)\,(1 + a\cos^2\varkappa)^{-1}, \quad J = \int_0^1 \Phi_3'\sin\theta\, dx \tag{5.6}$$

It is seen from (5. 3)—(5. 6) that J is real, hence $\gamma_2 = 0$. For $\beta = 1$

$$8J = (c_2 + 1)\ (\varkappa - 2\sin 2\varkappa + {}^3/_4\sin 4\varkappa) + 2\varkappa\,(1 - 2\cos 2\varkappa + {}^3/_4\cos 4\varkappa +$$
$$(16\varkappa)^{-1}\sin 4\varkappa) > 0$$

since for $\varkappa \leqslant {}^3/_4\, \pi$ each multiplier is positive, while for $\varkappa \geqslant {}^3/_4\, \pi$ with allowance for the increase of $c_2\,(\varkappa)$ we have

$$8J > (c_2 + 1)\ \ (\varkappa - {}^3/_4) - ({}^7/_2\, \varkappa + {}^1/_8) = f\,(\varkappa) \geqslant f\,({}^3/_4\ \ \pi) > 0$$

For $J \neq 0$ and $\lambda > 0$ Eq.(1.4) can have one stable stationary solution Q or two stable (with a considerable and a small $|\,Q\,|$) and one unstable.

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ON THE DETERMINATION OF THE SHAPE OF BODIES FORMED BY SOLIDIFICATION OF THE FLUID PHASE OF THE STREAM

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The problem of determination of the shape of bodies that have solidified in a moving fluid, of heat exchange between these and the fluid is encountered in domains such as underground construction by freezing water-saturated rocks, heat

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